ISRAEL JOURNAL OF MATHEMATICS, Vol. 25, 1976

COMPLETENESS THEOREMS FOR CONTINUOUS FUNCTIONS AND PRODUCT TOPOLOGIES

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ABSTRACT
In this paper we formulate a first order theory of continuous functions on product topologies via generalized quantifiers. We present an axiom system for continuous functions on product topologies and prove a completeness theorem for them with respect to topological models. We also show that if a theory has a topological model which satisfies the Hausdorff separation axiom, then it has a 0-dimensional, normal topological model. We conclude by obtaining an axiomatization for topological algebraic structures, e.g. topological groups, proving a completeness theorem for the analogue with countable conjunctions and disjunctions, and presenting counterexamples to interpolation and definability.

§0. Introduction

In [12] we developed a first order theory of topology using the notion of generalized quantifiers. In that paper we interpreted $Qx\varphi(x)$ to mean that the set defined by $\varphi(x)$ is "open". The main result was a proof of a completeness theorem for topology from the following natural set of axioms:

$Qx\ (x = x)$,

$Qx\ (x \neq x)$,

$Qx\varphi \wedge Qx\psi \rightarrow Qx(\varphi \wedge \psi)$,

$\forall y Qx\varphi(x, y) \rightarrow Qx \exists y\varphi(x, y)$.

This paper continues the study of first order topology by presenting a first order theory of continuous functions on product spaces. Our approach to product topologies is via generalized quantifiers. We add to the first order language, L, new quantifier symbols $Q^n x_1, \cdots, x_n$ for each $n \in \omega$. The intended interpretation of $Q^n x_1, \cdots, x_n\varphi(x_1, \cdots, x_n)$ is that the set defined by $\varphi(x_1, \cdots, x_n)$ is "open" in the nth product topology.

This formalization enables us to show in §2 the completeness of the theory of product spaces with continuous functions using the following natural formalization of the topological notions:

$Q^n x_1, \cdots, x_n \ (x_1 = x_1)$,
$Q^n x_1, \cdots, x_n \ (x_1 \neq x_1)$,
$Q^n x_1, \cdots, x_n \varphi \wedge Q^n x_1, \cdots, x_n \psi \rightarrow Q^n x_1, \cdots, x_n (\varphi \wedge \psi)$,
$\forall y Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n y) \rightarrow Q^n x_1, \cdots, x_n \ \exists y \varphi(x_1, \cdots, x_n, y)$,
$Q^n x_1, \cdots, x_n \varphi \wedge Q^m x_{m+1}, \cdots, x_{m+n} \psi \rightarrow Q^{n+m} x_1, \cdots, x_{m+n} \ (\varphi \wedge \psi)$,
$Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n) \rightarrow Q^k x_{i_1}, \cdots, x_{i_k} \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$,
where $\sigma : m \rightarrow m$, $, |\sigma[m]| = k$ and range $\sigma = \{i_1 < \cdots < i_k\}$;
$Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n) \rightarrow \forall x_1, \cdots, x_k \, Q^{n-k} x_{k+1}, \cdots, x_n \varphi(x_1, \cdots, x_n)$,
$Q^n y_1, \cdots, y_m \psi(y_1, \cdots, y_m) \rightarrow Q^{m+n-k} z_1, \cdots, z_m, y_{k+1}, \cdots, y_m$,
$(\exists y_1, \cdots, y_k \ (\psi(y_1, \cdots, y_m) \wedge \varphi(z_1, \cdots, z_n, y_1, \cdots, y_k)))$,

where $\varphi(x_1, \cdots, x_n, y_1, \cdots, y_m)$ defines an (n, k)-ary relation.

We show this by adding enough open sets to the topology to insure that every "open" set in the $Q^n x_1, \cdots, x_n$ interpretation is the union of open n-boxes.

In §3 we present several applications of the basic completeness theorem. The first is that any $L(Q^n_{n\in\omega})$ theory which satisfies $Q^2 xy \ (x \neq y)$, i.e. the Hausdorff separation axiom, has an interpretation where the topology is 0-dimensional and normal. One should notice the similarity of this result to a result in [12] where we showed that an $L(Q)$ theory which is consistent with $\forall y Q x \ (x \neq y)$ has a 0-dimensional normal topological model.

Other results include an $L(Q)$ axiomatization of the $L(Q)$ theories of topological groups and vector spaces, a completeness theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ and counterexamples to the interpolation and definability problems for $L(Q^n_{n\in\omega})$.

§1. Preliminaries

Take the first order predicate calculus L with the identity symbol $=$. We form the language $L(Q^n_{n\in\omega})$ by adding to L new quantifier symbols Q^n for $n \in \omega$. Thus $L(Q^n_{n\in\omega})$ has the quantifiers $(\exists x)$, $(\forall x)$, and $(Q^n x_1, \cdots, x_n)$ for $n \in \omega$. The set of formulas of $L(Q^n_{n\in\omega})$ is the smallest set which contains all the atomic formulas and is closed under $\wedge$, $\vee$, $\sim$, $(\exists x)$, $(\forall x)$ and $(Q^n x_1, \cdots, x_n)$ for $n \in \omega$. We will use the convention that $\varphi(v_1, \cdots, v_n)$ denotes a formula of $L(Q^n_{n\in\omega})$ whose free variables are among $v_1, \cdots, v_n$. Sentences are formulas without free variables.

Take $\mathfrak{A}$ to be a model of L and $\mathfrak{q}_n \subseteq S(A^n)$ and form $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$.

$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ is called a *weak model* for $L(Q^n_{n\in\omega})$. The notion of an k-tuple $a_1, \cdots, a_k \in A$ satisfying a formula $\varphi(v_1, \cdots, v_k)$ of $L(Q^n_{n\in\omega})$ in $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ is defined in the usual manner by induction on the complexity of φ and is denoted by

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_1, \mathfrak{q}_3, \cdots) \models \varphi[a_1, \cdots, a_k].$$

The $Q^n x_1, \cdots, x_n$ clause is defined as follows:

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models (Q^n v_m, \cdots, v_{m+n})\varphi[a_1, \cdots, v_m, \cdots, v_{m+n}, \cdots, a_k]$$

if and only if

$$\{\langle b_m, \cdots, b_{m+n}\rangle \mid (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$$
$$\models \varphi[a_1, \cdots, a_{m-1}, b_m, \cdots, b_{m+n}, \cdots, a_k]\} \in \mathfrak{q}_n,$$

where $\varphi(v_1, \cdots, v_{k+n})$ is a formula of $L(Q^n_{n\in\omega})$. The other clauses in the definition are the familiar ones for L. It is easy to check by induction on the complexity of φ that if all the free variables of $\varphi(v_1, \cdots, v_n)$ are among $v_1, \cdots, v_n$ and if $a_1 = b_1, \cdots, a_n = b_n$ then

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \varphi[a_1, \cdots, a_n]$$

if and only if

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \varphi[b_1, \cdots, b_n].$$

The axioms for $L(Q^n_{n\in\omega})$ are:

i) $\forall x_1, \cdots, x_n \forall x(\varphi \leftrightarrow \psi) \rightarrow (Q^n x_1, \cdots, x_n \varphi \leftrightarrow Q^n x_1, \cdots, x_n \psi)$,

ii) $Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n) \leftrightarrow Q^n y_1, \cdots, y_n \varphi(y_1, \cdots, y_n)$.

The rules of inference for $L(Q^n_{n\in\omega})$ are the same as for L, namely:

Modus Ponens: From φ, $\varphi \rightarrow \psi$ infer ψ.

Generalization: From φ infer $(\forall x)\varphi$.

For convenience we denote the sublogic $L(Q^1)$ of $L(Q^n_{n\in\omega})$ by $L(Q)$. A more explicit presentation of the $L(Q)$ version of the following theorems is found in Keisler [7]. We will not present the proofs for $L(Q^n_{n\in\omega})$ since they are analogous.

THEOREM 1.1. (Weak Completeness Theorem). *Σ is consistent in $L(Q^n_{n\in\omega})$ if and only if Σ has a weak model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$, where the elements of each $\mathfrak{q}_n$ are all $L(Q^n_{n\in\omega})$ definable.*

Let $L_{\omega_1\omega}$ be the infinitary logic with countable conjunctions and finitary quantification. Then $L_{\omega_1\omega}(Q^n_{n\in\omega})$ is the logic formed by adding to $L_{\omega_1\omega}$ the quantifier symbols $Q^n x_1, \cdots, x_n$ for $n \in \omega$.

More formally, the axioms and rules of inference for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ are just those for $L(Q)$ and $L_{\omega_1\omega}$. For the $L(Q)$ version of the following theorems see [7].

THEOREM 1.2 (Completeness Theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$). *A sentence φ of $L_{\omega_1\omega}(Q^n_{n\in\omega})$ is consistent if and only if φ has a weak model*

We now proceed to present several definitions and theorems which will be needed in this paper.

DEFINITION 1.3 (Tarski and Vaught). $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ is said to be an *elementary extension of* $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$, in symbols $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) < (\mathfrak{B}, r_1, r_2, r_3, \cdots)$, if and only if $A \subseteq B$ and for all formulas $\varphi(x_1, \cdots, x_n)$ of $L(Q^n_{n\in\omega})$ and all $a_1, \cdots, a_n \in A$ we have

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \varphi[a_1, \cdots, a_n]$$
$$\text{iff } (\mathfrak{B}, r_1, r_2, r_3, \cdots) \models \varphi[a_1, \cdots, a_n].$$

A sequence $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots)$, $\alpha < \gamma$, of weak models is said to be an *elementary chain* if and only if we have $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots) < (\mathfrak{A}_\beta, \mathfrak{q}^\beta_1, \mathfrak{q}^\beta_2, \mathfrak{q}^\beta_3, \cdots)$ for all $\alpha < \beta < \gamma$.

The union of an elementary chain $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots)$, $\alpha < \gamma$, is the weak model

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) = \bigcup_{\alpha<\gamma} (\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots)$$

such that $\mathfrak{A} = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ and $\mathfrak{q}_n = \{S \subset A^n \mid \text{for some } \beta < \gamma,\ \beta \leqq \alpha < \gamma \text{ implies } S \cap A^n_\alpha \in \mathfrak{q}^\alpha_n\}$.

These definitions enable us to state:

THEOREM 1.4. *Let $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots)$, $\alpha < \gamma$, be an elementary chain and let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be the union. Then for all $\alpha < \gamma$,*

$$(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha_1, \mathfrak{q}^\alpha_2, \mathfrak{q}^\alpha_3, \cdots) < (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots).$$

We now present the last model theoretic theorem needed for weak models.

THEOREM 1.5 (Löwenheim Skolem Theorem).

a) *Let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be a weak model of $L(Q^n_{n\in\omega})$ and $\aleph$ a cardinal such that $|L| \leqq \aleph \leqq |A|$. Then there is a weak model $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ such that $(\mathfrak{B}, r_1, r_2, r_3, \cdots) < (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ and $|B| = \aleph$.*

b) *Let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be a weak model of $L(Q^n_{n\in\omega})$ and $\aleph$ a cardinal such* *that* $|L|+|A|\leqq\aleph$. *Then there is a weak model* $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ *such that* $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) < (\mathfrak{B}, r_1, r_2, r_3, \cdots)$ *and* $|B| = \aleph$.

For notational convenience, if $\vec{x} = \langle x_1, \cdots, x_n\rangle$ is a set of distinct variables and $\vec{z} = \langle z_1, \cdots, z_n\rangle$ is a set of elements of some model, then $\vec{t} \in (\vec{x}/\vec{z})$ iff for each $1 \leqq i \leqq n$, t_i is either x_i or z_i. Thus $\varphi(\vec{t})$ has the obvious interpretation. Also we will regard an (m, n)-ary relation (or function) as a formula $\varphi(x_1, \cdots, x_m, y_1, \cdots, y_n)$ which defines it as a relation.

In order to study our first order topology we will now give the necessary basic definitions and theorems.

Topological notions are standard as in [2]. Recall that a topology generated by a set, σ, of subsets of a space is the collection of arbitrary unions of finite intersections of σ. A collection of generators for the product topology of $\Phi = \Pi_{\mu\in M} X_\mu$ is $\{\{f \in \Phi \mid f(\mu) \in U\} \mid U \text{ open in } X_\mu,\ \mu \in M\}$. If $X = X_\gamma$, $\gamma \in M$ then ΠX_μ is called the topological power of X.

Let $f: X \to Y$ be continuous. Then

$$f^{-1}(Y - B) = X - f^{-1}(B);\quad f^{-1}\Big(\bigcup_{\alpha\in M} E_\alpha\Big) = \bigcup_{\alpha\in M} f^{-1}(E_\alpha);$$

$$f^{-1}\Big(\bigcap_{\alpha\in M} E_\alpha\Big) = \bigcap_{\alpha\in M} f^{-1}(E_\alpha).$$

Concluding the basic definitions and theorems we present the following basic model theoretic definitions.

DEFINITION 1.7. A weak $L(Q^n_{n\in\omega})$ model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ is called *topological* iff each $\mathfrak{q}_i$, $i \in \omega$, is a topology on A^i. (Notice that an $L(Q)$ model, $(\mathfrak{A}, \mathfrak{q})$, is called topological iff $\mathfrak{q}$ is a topology on A.)

Definition 1.8 now enables us to state the final definition of this section.

DEFINITION 1.8. A topological model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ is called *complete* iff $\mathfrak{q}_k$ is the kth topological product of $\mathfrak{q}_1$ on A. For notational convenience, if $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ is a complete topological model we abbreviate it by just writing $(\mathfrak{A}, \mathfrak{q})$, and the remaining structure is understood.

§2. The basic completeness theorem

We will show in this section that the theory of continuous functions (relations) on product spaces has the following axiomatization.

(B0) All axiom schemes for $L(Q^n_{n\in\omega})$.

(B1) $Q^n x_1, \cdots, x_n\ (x_1 = x_1)$.
(B2) $Q^n x_1, \cdots, x_n\ (x_1 \neq x_1)$.
(B3) $Q^n x_1, \cdots, x_n \varphi \wedge Q^n x_1, \cdots, x_n \psi \rightarrow Q^n x_1, \cdots, x_n\ (\varphi \wedge \psi)$.
(B4) $\forall y Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n, y) \rightarrow Q^n x_1, \cdots, x_n\ \exists y \varphi(x_1, \cdots, x_n, y)$.
(B5) $Q^n x_1, \cdots, x_n \varphi \wedge Q^m x_{n+1}, \cdots, x_{n+m} \psi \rightarrow Q^{n+m} x_1, \cdots, x_{n+m}\ (\varphi \wedge \psi)$.
(B6) $Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n) \rightarrow Q^k x_{i_1}, \cdots, x_{i_k} \varphi(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$,
where $\sigma : m \rightarrow m$, $|\sigma[m]| = k$ and the range of $\sigma = \{i_1 <, \cdots, < i_k\}$.
(B7) $Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n) \rightarrow \forall x_1, \cdots, x_k Q^{n-k} x_{k+1}, \cdots, x_n \varphi(x_1, \cdots, x_n)$.
(B8)$_\varphi$ $Q^m y_1, \cdots, y_m \psi(y_1, \cdots, y_m) \rightarrow Q^{m+n-k} z_1, \cdots, z_n y_{k+1}, \cdots, y_m$

$$(\exists y_1, \cdots, y_k (\psi(y_1, \cdots, y_m) \wedge \varphi(z_1, \cdots, z_n, y_1, \cdots, y_k))),$$

where $\varphi(z_1, \cdots, z_n, y_1, \cdots, y_k)$ defines an (n, k)-ary relation.

Axioms B0–B4 formalize our notion of a topology and are the $L(Q^n_{n\in\omega})$ analogues of the axioms used by the author in [12] to show a completeness theorem for topological models. The meaning of B5 is that the product of open sets is open. B6 and B7 state that the permutation, projection, or consolidation of an open set is open. Finally, B8$_\varphi$ says that φ defines a continuous relation, i.e., the inverse image of a slice of an open set is open.

One can see, without much difficulty, by using the definition of a product topology and continuous functions that B0–B8$_\varphi$ are true in every complete-topological model where φ is continuous. The converse of this is our completeness theorem and is stated as follows:

If Σ is an $L(Q^n_{n\in\omega})$ theory and φ_α, $\alpha \in I$, are (n_α, m_α)-ary relations for $\alpha \in I$, then Σ has a complete topological model, where each relation φ_α, $\alpha \in I$, is continuous if and only if Σ is consistent with B0, $\cdots$, B7 *and* B8$_{\varphi_\alpha}$ *for $\alpha \in I$.*

Before we proceed to prove this we need several basic facts from topology and a theorem of the author [12].

Let (X, τ) be a topological space. Then if $Y \subseteq X$ and Y is not open, there is a $c_y \in Y$ such that if $\mathcal{O} \in \tau$ and $\mathcal{O} \subseteq Y$ then $c_y \notin \mathcal{O}$. In other words, any non-open set has at least one point which is not in any open subset of it.

Using this fact we proved in [12] the $L(Q)$ analogue of the following completeness theorem.

THEOREM 2.1. *Let Σ be an $L(Q^n_{n\in\omega})$ theory. Then Σ is consistent with* B0–B4 *if and only if Σ has a topological model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ where each $\mathfrak{q}_n$, $n \in \omega$, is generated by the definable Q^n open sets (with parameter from A) and $|\mathfrak{A}| \geq |L|$.*

Suppose we have a topological model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ where each $\mathfrak{q}_n$, $n \in \omega$, is generated by the Q^n definable open sets and $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ models B0–B7. Also assume φ_α, $\alpha \in I$, are definable relations of $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ which are "continuous", i.e. $B8_{\varphi_\alpha}$, $\alpha \in I$, holds. Then assume we are given $\{\mathcal{U}_\beta\}_{\beta \in D}$, a collection of subsets of A.

If we add the $\{\mathcal{U}_\beta\}_{\beta \in D}$ to $\mathfrak{q}_1$ and still expect to have a model satisfying (in the expanded language with a U_β for each $\mathcal{U}_\beta$) B0–B7 and $B8_{\varphi_\alpha}$, $\alpha \in I$, what do we need to add to q_n, $n \in \omega$? The following is the answer.

Let φ_α, $\alpha \in I$, be a collection of (n_α, m_α)-ary relations which satisfy $B8_{\varphi_\alpha}$, $\alpha \in I$. Let $\varphi_\alpha^{-1} = \{(\vec{b}, \vec{a}) \mid (\vec{a}, \vec{b}) \in \varphi_\alpha\}$ be the inverse relation of φ_α, $\alpha \in I$. We then define a collection of (definable) relations as follows.

$$WT_0 = \{\varphi_\alpha^{-1}\}_{\alpha \in I} \cup \{\text{identity relation on each } A^n, n \in \omega\}$$

$$\begin{aligned} WT_{n+1} = WT_n \; & \cup \{\varphi(x_{\sigma(1)}, \cdots, x_{\sigma(n_\varphi)}, y_1, \cdots, y_{m_\varphi}) \ \sigma \text{ maps } n_\varphi \text{ into } n_\varphi \text{ where } \varphi \in WT_n\} \\ & \cup \{\varphi(x_1, \cdots, x_{n_\varphi}, y_{\sigma(1)}, \cdots, y_{\sigma(m_\varphi)}) \ \sigma \text{ maps } m_\varphi \text{ into } m_\varphi \text{ where } \varphi \in WT_n\} \\ & \cup \{\varphi(x_1, \cdots, x_{n_\varphi}, y_1, \cdots, y_{m_\varphi}) \wedge \psi(z_1, \cdots, z_{n_\psi}, t_1, \cdots, t_{m_\psi}) \text{ where } \varphi, \psi \in WT_n\} \\ & \cup \{\varphi(x_1, \cdots, c, \cdots, x_{n_\varphi}, y_1, \cdots, y_{m_\varphi}) \text{ where } \varphi \in WT_n \text{ and } c \text{ an individual constant symbol}\} \\ & \cup \{\varphi(x_1, \cdots, x_{n_\varphi}, y_1, \cdots, c, \cdots, y_{m_\varphi}) \text{ where } \varphi \in WT_n \text{ and } c \text{ an individual constant symbol}\} \\ & \cup \{(\exists y_1, \cdots, y_k\, (\varphi(x_1, \cdots, x_{n_\varphi}, y_1, \cdots, y_{m_\varphi}) \\ & \qquad \wedge \psi(y_1, \cdots, y_{n_\psi}, z_1, \cdots, z_{m_\psi}))), \quad k \leqq m_\varphi, k \leqq m_\psi, \\ & \qquad \text{i.e. the composition of the two relations, where } \varphi, \psi \in WT_n\}. \end{aligned}$$

Let $WT = \bigcup_{n \in \omega} WT_n$.

The intuitive meaning of WT is that it is the smallest collection of definable relations containing WT_0 and closed under composition, projection, products and mappings of the variables. Hence, since each φ_α satisfies $B8_{\varphi_\alpha}$, $\alpha \in I$ and $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ models B0–$B8_{\varphi_\alpha}$, $\alpha \in I$, we have that each $\varphi \in WT$ takes definable open sets to definable open sets.

Now define $\mathfrak{q}_n^*$, $n \in \omega$, as follows:

$\mathfrak{q}_n^*$ = the topology generated by $\{\varphi(\prod_{j=1}^k B_j) \mid$ where each $B_j \in \mathfrak{q}_{k_j}$ or $B_j = \mathcal{U}_{\beta_j}$, for $1 \leqq j \leqq k$, $\varphi \in WT$ and φ maps into $A^n\}$.

LEMMA 2.2. *Let* $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ *be as above. Then* $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ *models* B0–B7 *and* $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$, *in the expanded language containing a* U_β *for each* $\mathcal{U}_\beta$.

PROOF. Since each $\mathfrak{q}_n^*$, $n \in \omega$, is a topology we have that $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ models B0–B4. To show B5 let us take $\mathcal{O}_1^* \in \mathfrak{q}_k^*$ and $\mathcal{O}_2^* \in \mathfrak{q}_l^*$. Now $\mathcal{O}_1^* = \bigcup_{\alpha \in E} \mathcal{O}_\alpha^*$ and $\mathcal{O}_2^* = \bigcup_{\beta \in D} \mathcal{O}_\beta^*$ and $\mathcal{O}_1^* \times \mathcal{O}_2^* = \bigcup_{(\alpha,\beta) \in E \times D} \mathcal{O}_\alpha^* \times \mathcal{O}_\beta^*$ where each $\mathcal{O}_\alpha^*$, $\mathcal{O}_\beta^*$ is in the generating set. Since $\mathcal{O}_\alpha^*$, $\mathcal{O}_\beta^*$ are in the generating set, $\mathcal{O}_\alpha^* = \bigcap_{i=1}^{s_\alpha} \varphi_{\alpha_i}(\Pi B_j)$ and $\mathcal{O}_\beta^* = \bigcap_{i=1}^{s_\beta} \varphi_{\beta_i}(\Pi B_j')$. Thus

$$\mathcal{O}_\alpha^* \times \mathcal{O}_\beta^* = \cap \varphi_\delta ((\Pi B_i) \times (\Pi B_j'))$$

where $\varphi_\delta \in WT$. This uses the fact that the product of intersections is the intersection of a product. Hence, $\mathcal{O}_\alpha^* \times \mathcal{O}_\beta^* \in \mathfrak{q}_{k+l}^*$.

If $\mathcal{O}^* = \bigcup_{\alpha \in I} \mathcal{O}_\alpha^* \in \mathfrak{q}_n^*$, and $\sigma : m \to m, |\sigma(m)| = k$ and rang $\sigma = \{i_1 < \cdots < i_k\}$, then

$$\mathcal{O}^{*\sigma} = \{\langle t_{\sigma(1)}, \cdots, t_{\sigma(n)} \rangle \mid \langle t_1, \cdots, t_n \rangle \in \mathcal{O}^*\}$$

$$= \bigcup_{\alpha \in I} (\mathcal{O}_\alpha^*)^\sigma \in \mathfrak{q}_k^*$$

since σ applied to the identity map on n is in WT and $(\cap \mathcal{O}^*)^\sigma = \cap (\mathcal{O}^{*\sigma})$. Hence we have B6. B7 follows similarly, since the projection of the identity is also in WT and a projection of an intersection is the intersection of a projection.

Given the relation φ_α we need to show $\mathrm{B8}_{\varphi_\alpha}$ holds in $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$. Suppose $\mathcal{O}^* = \bigcup_{\beta \in D} \mathcal{O}_\beta^* \in \mathfrak{q}_n^*$, $\mathcal{O}_\beta^*$ a basic open set of $\mathfrak{q}_n^*$, i.e., $\mathcal{O}_\beta^* = \cap \varphi_{\beta_i}(\Pi B_j)$. Then

$$\mathcal{O}^{*\varphi_\alpha} = \{\langle \varphi_\alpha(\langle t_1, \cdots, t_k \rangle), t_{k+1}, \cdots, t_n \rangle \mid \text{ where } \langle t_1, \cdots, t_n \rangle \in \mathcal{O}^*\}$$

$$= \bigcup_{\beta \in D} (\mathcal{O}_\beta^*)^{\varphi_\alpha} \in \mathfrak{q}_{n+m-k}^*$$

by construction of WT and the fact that

$$\mathcal{O}_\beta^{*\varphi_\alpha} = (\cap \varphi_\beta(\Pi B_j))^{\varphi_\alpha} = \cap (\varphi_\beta(\Pi B_j)^{\varphi_\alpha}) = \cap \varphi_\delta (\Pi B_j).$$

Thus we have shown the lemma.

Now we will proceed to prove the main completeness theorem by presenting the following lemma which tells us that for each $\vec{c} \in \mathcal{O}$ (an open set in the Q^n interpretation) we can add a $\Pi_{i=1}^n \mathcal{V}_i$ (an open n-box) to the Q^n interpretation such that $\vec{c} \in \Pi_{i=1}^n \mathcal{V}_i \subseteq \mathcal{O}$ and still keep B0–$\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$.

LEMMA 2.3. *Let Σ be an $L(Q^n_{n\in\omega})$ theory consistent with* B0–B7 *and* $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. *If* $\vec{c} = \langle c_1, \cdots, c_n\rangle$ *is an n-tuple so that* $\varphi(c_1, \cdots, c_n)$ *and* $Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n)$ *are consistent with* Σ *then* $V_i(c_i)$, $QxV_i(x)$, $1 \leqq i \leqq n$ *and* $\forall x_1, \cdots, x_n (\wedge_{i=1}^n V(x_i) \rightarrow \varphi(x_1, \cdots, x_n))$ *is consistent with* Σ *and* $\mathrm{B0}$–$\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. *Here the* $V_i(x)$, $1 \leqq i \leqq n$, *are new one place predicate symbols.*

PROOF. We need only prove this for countable Σ since then, by using Theorem 1.1, we have it for all Σ and $\mathrm{B0}, \cdots, \mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. Let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be a topological model of Σ generated by the definable open sets. This is possible since Σ is consistent with B0–B4 and Theorem 2.1.

We want to define $\mathcal{V}_i \subseteq A$, $1 \leqq i \leqq n$ such that $\vec{c}^{\mathfrak{A}} \in \Pi_{i=1}^n \mathcal{V}_i \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$ and forming $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ from the $\{\mathcal{V}_i\}_{1 \leqq i \leqq n}$ we have that

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \underset{L(Q^n_{n\in\omega})(A)}{\equiv} (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots).$$

To do this we will construct the $\mathcal{V}_i$'s by induction.

Suppose we have picked $z_i^1, \cdots, z_i^k$ for each $\mathcal{V}_i$ so that

$$\prod_{i=1}^n \{z_i, \cdots, z_i^k\} \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$$

and $\langle c_1^{\mathfrak{A}}, \cdots, c_n^{\mathfrak{A}}\rangle = \langle z_1^1, \cdots, z_n^1\rangle$. Now to pick the z_i^{k+1}, $1 \leqq i \leqq n$ we want to insure that

$$\prod_{i=1}^n \{z_i^1, \cdots, z_i^k, z_i^{k+1}\} \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$$

and also to somehow insure that if $\sigma(x_1, \cdots, x_m)$ is a formula of $L(Q^n_{n\in\omega})$ so that

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \sim Q^m x_1, \cdots, x_m \sigma(x_1, \cdots, x_m)$$

then we do not get $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \models Q^m x_1, \cdots, x_m \sigma(x_1, \cdots, x_m)$. That is

$$[\sigma(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)} \neq \bigcup_{\beta \in D} \mathcal{O}_\beta^*$$

where $\mathcal{O}_\beta^* \in \mathfrak{q}_m^*$ and is a basic open set.

We will then assume that we have some countable enumeration of basic open sets, i.e. $\mathcal{O}_\beta = \cap\, \varphi_{\beta_i} (\Pi B_j)$, and $\sigma(x_1, \cdots, x_m)$ as above. We claim that the z_i^{k+1} can be picked such that

$$(*) \qquad \prod_{i=1}^n \{z_i^1, \cdots, z_i^k, z_i^{k+1}\} \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$$

and

$$(**) \qquad \text{if } \mathcal{O}^*_\beta \subseteq [\sigma(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$$

then there is an $\mathcal{O}_\beta$ such that

$$\mathcal{O}^*_\beta \subseteq \mathcal{O}_\beta \subseteq [\sigma(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$$

and $\mathcal{O}_\beta \in \mathfrak{q}_m$.

This is done as follows. To have (*) we must have

$$\langle z_1^{k+1},\cdots,z_n^{k+1}\rangle \in C = \bigcap_{\vec{z}\in\times_{i=1}^n\{z_i^1,\cdots,z_i^k\}} \ \bigcap_{\substack{\vec{t}\in(\vec{x}/\vec{z})\\ \vec{t}\neq\vec{z}}} \{\varphi(\vec{t})\}^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$$

where $\{\varphi(\vec{t})\}^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)} = \{(a_1,\cdots,a_n) \mid (\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)\models \varphi(k_1,\cdots,k_n)$ where $k_i = a_i$ if $t_i = x_i$, $k_i = z_i$ otherwise$\}$.

This follows, since if

$$\vec{z} = \langle z_1,\cdots,z_n\rangle \in \prod_{i=1}^n \{z_i^1,\cdots,z_i^{k+1}\} \subseteq [\varphi(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$$

then $\vec{z} \in \{\varphi(\vec{t})\}^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$. Conversely, suppose that we have $\vec{z}\in C$, then given any $\vec{z}_2 \in \Pi_{i=1}^n\{z_i^1,\cdots,z_i^k\}$ and any $\vec{t}\in(\vec{z}/\vec{z}_2)$ we have

$$\vec{t} \in [\varphi(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}.$$

To obtain (**) we can assume without loss of generality that $\mathcal{O}^*_\beta = \bigcap \varphi_{\beta_i}(\Pi_{j=1}^{s_i} B_j)$ and $B_j = \mathcal{V}_j$ for $1 \leqq j \leqq n$ and $B_i = \mathcal{O}_{\delta_i} \in \mathfrak{q}_m$ for $i > n$. Thus consider $\bigcap \varphi_{\beta_i}(C \times \Pi_{j=m}^{s_i}\mathcal{O}_{\delta_j})$. If

$$\bigcap \varphi_{\beta_i}\left(C \times \prod_{j=m}^{s_i} \mathcal{O}_{\delta_j}\right) - [\sigma(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)} \neq \varnothing,$$

then let $\langle z_1^{k+1},\cdots,z_n^{k+1}\rangle \in C$ so that

$$\left(\bigcap \varphi_{\beta_i}\left(\langle z_1^{k+1},\cdots,z_n^{k+1}\rangle \times \prod_{j=m}^{s_i}\mathcal{O}_{\delta_j}\right)\right) - [\sigma(x_1,\cdots,x_n)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)} \neq \varnothing.$$

Otherwise let $z_i^{k+1} = z_i^k$, $1 \leqq i \leqq n$ and we get that $\bigcap \varphi_{\beta_i}(C \times \Pi_{j=m}^{s_i}\mathcal{O}_{\delta_j}) \in \mathfrak{q}_m$ and is a subset of $[\sigma(x_1,\cdots,x_m)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$ which suffices since $\Pi_i\{z_i^1,\cdots,z_i^k\}\subseteq C$.

Now we will show that if $\mathcal{V}_i = \{z_i^k \mid k \in \omega\}$ then we have the conclusion to the lemma.

$$\vec{c} \in \prod_{i=1}^{n} \mathcal{V}_i \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$$

by (*). To show

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*) \underset{L(Q^n_{n\in\omega})(A)}{\equiv} (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$$

we use induction on the Q^m quantifiers. The only difficult case is the Q^m clause.

Since $\mathfrak{q}_i \subseteq \mathfrak{q}_i^*$ for all i we have that if

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m)$$

then

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m).$$

So suppose that

$$(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \sim Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m)$$

and

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m).$$

Hence

$$[\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)} = [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)}$$
$$= \bigcup_{\beta \in D} \mathcal{O}_\beta^*;$$

$\mathcal{O}_\beta^*$, $\beta \in D$, a basic open set of $\mathfrak{q}_m^*$. Thus by (**) each $\mathcal{O}_\beta^* \subseteq \mathcal{O}_\beta \subseteq [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$ and $\mathcal{O}_\beta \in \mathfrak{q}_m$. So $\bigcup_{\beta \in D} \mathcal{O}_\beta = \bigcup_{\beta \in D} \mathcal{O}_\beta^* = [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$, which is a contradiction. Hence the lemma is shown.

Now we are able to prove the basic completeness theorem.

THEOREM 2.4. *Let Σ be an $L(Q^n_{n\in\omega})$ theory. Then Σ is consistent with* B0, $\cdots$, B7 *and* $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$, *if and only if Σ has a complete topological model* $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ *such that each φ_α, $\alpha \in I$, is continuous.*

PROOF. (if direction) Straightforward since B0, $\cdots$, B7 and $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$ are true in every complete topological model where the φ_α, $\alpha \in I$, are continuous.

(only if direction) Assume Σ is consistent with B0, $\cdots$, B7 and $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. Then by Theorem 2.1 we have a topological model $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ of Σ and B0, $\cdots$, B7 and $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$ such that $|\mathfrak{A}| \geqq |L|$. By repeated applications of

Lemma 2.3 to the complete theory of $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ we obtain a topological model $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ of Σ, B0, $\cdots$, B7 and $B8_{\varphi_\alpha}$, $\alpha \in I$, such that $|\mathfrak{B}| = |\mathfrak{A}|$ and if

$$\vec{b} \in [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{B}, r_1, r_2, r_3, \cdots)} \in r_n$$

then there is a $\mathcal{V}_1, \cdots, \mathcal{V}_n \in r_1$ such that

$$\vec{b} \in \prod_{i=1}^{n} \mathcal{V}_i \subseteq [\varphi(x_1, \cdots, x_n)]^{(\mathfrak{B}, r_1, r_2, r_3, \cdots)}.$$

Hence $(\mathfrak{B}, r_1, r_2, r_3, \cdots)$ is complete. Notice that by Theorem 1.5 we can take $|\mathfrak{A}| = \aleph$ for any $\aleph \geqq |L|$ and thus $|\mathfrak{B}| = |\mathfrak{A}| = \aleph$.

If we omit $B8_{\varphi_\alpha}$, $\alpha \in I$ then we obtain the following interesting corollary.

Corollary 2.5. *Let Σ be an $L(Q^n_{n \in \omega})$ theory. Then Σ is consistent with* B0, $\cdots$, B7 *if and only if Σ has a complete topological model.*

Proof. This is a direct application of Theorem 2.4.

Corollary 2.6. (Compactness Theorem). *Let Σ be an $L(Q^n_{n \in \omega})$ theory. Then Σ has a complete topological model where each φ_α, $\alpha \in I$, is continuous if and only if every finite subset of Σ has a complete topological model where φ_α, $\alpha \in I$, is continuous.*

Proof. An easy application of the basic completeness theorem.

Corollary 2.7. *The set of $L(Q^n_{n \in \omega})$ sentences valid in every complete topological model (with φ_α, $\alpha \in I$, continuous) is recursively enumerable* in the language.

Proof. Theorem 2.4 shows that a sentence is provable from B0, $\cdots$, B7, and $B8_{\varphi_\alpha}$, $\alpha \in I$, if and only if it is valid, so we are done.

We can now prove a Löwenheim Skolem Theorem for complete topological models with continuous functions using the methods of Theorem 2.4 and [12].

Theorem 2.8.

a) *Let $(\mathfrak{A}, \mathfrak{q})$ be a complete topological model where each φ_α, $\alpha \in I$, is continuous. Then for any $\aleph \geqq |L| + |A|$ there is a complete topological model $(\mathfrak{B}, r)$ such that $(\mathfrak{A}, \mathfrak{q}) < (\mathfrak{B}, r)$, $|B| = \aleph$, and each φ_α is continuous in $(\mathfrak{B}, r)$.*

b) *Let $(\mathfrak{A}, \mathfrak{q})$ be a complete topological model where each φ_α, $\alpha \in I$, is continuous. Then for any $|L| \leqq \aleph \leqq |A|$ there is a complete topological model $(\mathfrak{B}, r) < (\mathfrak{A}, \mathfrak{q})$ such that $|B| = \aleph$, and each φ_α, $\alpha \in I$, is continuous in $(\mathfrak{B}, r)$.*

Proof.

a) By the methods of Theorem 2.4 (Completeness Theorem) and the remark at the end of its proof we can find a complete topological model $(\mathfrak{A}, \mathfrak{q}) < (\mathfrak{B}, r)$ such that $|B| = \aleph$ and each φ_α, $\alpha \in I$, is continuous in $(\mathfrak{B}, r)$.

b) Let f_i^φ, φ a formula of $L(Q^n_{n\in\omega})$, be as in the author's paper [12]. That is, for each formula, $\varphi(x_1, \cdots, x_n, y_1, \cdots, y_m)$, of $L(Q^n_{n\in\omega})$ we add a new function symbol $f_i^\varphi(y_1, \cdots, y_m)$ for $1 \leqq i \leqq n$ and given any formula

$$\psi(x_1, \cdots, x_n, z_1, \cdots, z_k)$$

let ψ^φ be

$$\begin{aligned} &\forall y_1, \cdots, y_m, z_1, \cdots, z_k\, (Q^n x_1, \cdots, x_n)\psi(x_1, \cdots, x_n, z_1, \cdots, z_k) \wedge \\ &\quad \sim Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n, y_1, \cdots, y_m) \wedge . \\ &\forall x_1, \cdots, x_n\, (\psi(x_1, \cdots, x_n, z_1, \cdots, z_k) \rightarrow \varphi(x_1, \cdots, x_n, y_1, \cdots, y_m)) \\ &\quad \rightarrow \varphi\, (f_1^\varphi\,(y_1, \cdots, y_n), \cdots, f_n^\varphi\,(y_1, \cdots, y_n), y_1, \cdots, y_n) \wedge \\ &\quad \sim \psi(f_1^\varphi(y_1, \cdots, y_n), \cdots, f_n^\varphi(y_1, \cdots, y_n), z_1, \cdots, z_k)). \end{aligned}$$

That is, if φ defines a non-open set in A^n then $\langle f_1^\varphi, \cdots, f_n^\varphi \rangle$ is a point of φ which is not in any open subset of it. (This was used in the proof of Theorem 2.1, confer [12].) Since $\mathfrak{A}$ is a complete topological model where each φ_α, $\alpha \in I$, is continuous we can expand L to an L' and $\mathfrak{A}$ to an $\mathfrak{A}'$ so that for every $\varphi(x_1, \cdots, x_n, y_1, \cdots, y_m) \in L'(Q^n_{n\in\omega})$ we have that there are $V_i^\varphi(z_i, x_i, y_1, \cdots, y_n)$, $1 \leqq i \leqq n$, such that

$$\begin{aligned} \varphi^V = &\forall x_1, \cdots, x_n, \forall y_1, \cdots, y_m\, (Q^n x_1, \cdots, x_n \varphi(x_1, \cdots, x_n, y_1, \cdots, y_m) \\ &\rightarrow (\textstyle\bigwedge_{i=1}^n V_i^\varphi(x_i, x_i, y_1, \cdots, y_n)) \wedge \forall z_1, \cdots, z_n\, (\textstyle\bigwedge_{i=1}^n V_i^\varphi(z_i, x_i, y_1, \cdots, y_n) \\ &\rightarrow \varphi(z_1, \cdots, z_n, y_1, \cdots, y_n)) \wedge (\textstyle\bigwedge_{i=1}^n Qz_i V_i^\varphi(z_i, x_i, y_1, \cdots, y_n)))) \end{aligned}$$

holds.

This says that each point in a φ definable open subset has a definable n-box around it contained in the open set defined by φ. Thus let $\mathfrak{A}^* = (\mathfrak{A}', f^\varphi_{\varphi\in L'(Q^n_{n\in\omega})})$. Then $(\mathfrak{A}^*, \mathfrak{q})$ has an elementary submodel $(\mathfrak{B}, r)$, $|B| = \aleph$, by the Löwenheim Skolem Theorem for weak models. Hence if r^* is the topology generated by the $L'(Q^n_{n\in\omega})$ definable elements (with parameters) of r we are done.

Note that if we let

$$\kappa((\mathfrak{A}, \mathfrak{q})) = \inf\{|\beta| : \beta \text{ is a basis for } \mathfrak{q}\}$$

then we have shown that we can obtain a model, $(\mathfrak{B}, r^*)$, where $\kappa((\mathfrak{B}, r^*)) \leqq |\mathfrak{B}|$.

§ 3. Applications

In this section we will present several applications of the completeness theorems and the techniques used in their proof.

§3.1 contains the proof that the $L(Q^n_{n\in\omega})$ theory of 0-dimensional normal (paracompact) complete topological models is equivalent to the $L(Q^n_{n\in\omega})$ theory generated by $Q^2xy\ (x\neq y)$ which is a logical formulation of the Hausdorff separation axiom. As in [12], if we apply this to countable theories we obtain a metrizable model.

In §3.2 we show that we can extend $L(Q)$ theories with "coordinatewise – continuous" functions to $L(Q^n_{n\in\omega})$ theories where the functions are continuous. As a corollary to this theorem we present an $L(Q)$ axiomatization of the theory of topological groups and vector spaces.

Concluding this section we prove a completeness theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ which is formed by combining $L(Q^n_{n\in\omega})$ with $L_{\omega_1\omega}$.

§3.1 We will now present several definitions and theorems from topology which will permit us to present the main theorem of this section.

Definition 3.1.1. A topological space is called *Hausdorff* if every pair of distinct points can be separated by disjoint open sets.

Definition 3.1.2. A topological space is called *regular* if each point and disjoint closed set can be separated by disjoint open sets. (We assume that points are closed.)

Definition 3.1.3. A topological space is called *normal* if every pair of disjoint closed sets can be separated by disjoint open sets. (Again assume points are closed.)

Definition 3.1.4. A topological space is called 0-*dimensional* if its topology is generated by sets which are both open and closed (clopen).

Let (X,τ) be a topological space. If we define the diagonal of X, in symbols $\Delta(X)$, to be $\{\langle x_1, x_2\rangle \mid x_1 = x_2\}$ then we can show the following topological result.

A topological space (X,τ) is Hausdorff if and only if the diagonal of X is closed (in X^2).

This now enables us to state and prove that the $L(Q^n_{n\in\omega})$ theory of 0-dimensional normal complete topological models is the same as the $L(Q^n_{n\in\omega})$ theory of Hausdorff models. One should note that the Hausdorff separation axiom is equivalent by the above remark to $Q^2xy\ (x\neq y)$.

We first prove the following important lemma which is analogous to Lemma 2.3.

LEMMA 3.1.5. *Let Σ be an $L(Q^n_{n\in\omega})$ theory consistent with* B0–B7 *and* $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. *Then if Σ is consistent with $Q^2xy\ (x \neq y)$, $Qx \sim \psi(x)$, $Qx \sim \varphi(x)$, and $\sim \exists x(\psi(x) \wedge \varphi(x))$ (i.e. ψ and φ define disjoint closed sets) then $\forall x(\psi(x) \rightarrow U^{\psi,\varphi}(x))$, $\forall x(\varphi(x) \rightarrow \sim U^{\psi,\varphi}(x))$, $QxU^{\psi,\varphi}(x)$, and $Qx \sim U^{\psi,\varphi}(x)$ are consistent with Σ,* B0, $\cdots$, B7 *and* $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$. *Here $U^{\psi,\varphi}(x)$ is a new one place predicate symbol. The conclusion means that $U^{\psi,\varphi}$ and $\sim U^{\psi,\varphi}$ define open sets which separate the sets defined by ψ and φ.*

PROOF. Again we only need to show the lemma for countable Σ; then using the compactness theorem, we obtain it for all Σ. Thus let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be a countable topological model of Σ, B0, $\cdots$, B7 and $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$, where the $\mathfrak{q}_i$ are generated by the definable open sets.

Again as in Lemma 2.3 we want to form $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ from a $\mathcal{U}$, $A - \mathcal{U}$ and $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ such that $[\psi]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)} \subseteq \mathcal{U}$ and $[\varphi]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)} \subseteq A - \mathcal{U}$ and

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \equiv_{L(Q^n_{n\in\omega})(A)} (\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots).$$

Also, as in Lemma 2.3 we will define $\mathcal{U}$ and $A - \mathcal{U}$ by induction. Thus suppose we have defined $r_1, \cdots, r_k$ for $\mathcal{U}$ and $s_1, \cdots, s_k$ for $A - \mathcal{U}$. Now we will define an r_{k+1} for $\mathcal{U}$ and an s_{k+1} for $A - \mathcal{U}$ so that

$$[\sigma(x_1, \cdots, x_n)]^{(\mathfrak{A},\mathfrak{q}_1^*,\mathfrak{q}_2^*,\mathfrak{q}_3^*,\cdots)} \neq \bigcup_{\beta\in D} \mathcal{O}^*_\beta,$$

$\mathcal{O}^*_\beta$ a basic open set of $\mathfrak{q}^*_m$.

We, again as in Lemma 2.3, assume that we have a countable enumeration of the potential basic open sets, i.e. $\varphi_\alpha\ (\Pi_{i=1}^s B_i)$, and the $\sigma(x_1, \cdots, x_m)$, defining non-open sets. It is claimed that r_{k+1} and s_{k+1} can be picked so that

$$(*) \qquad r_{k+1}, s_{k+1} \not\in \{r_1, \cdots, r_k, s_1, \cdots, s_k\}$$
$$\text{and } r_{k+1} \neq s_{k+1} \in A - [\varphi \vee \psi]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}.$$

r_{k+1} and s_{k+1} also have the property that if $\mathcal{O}^*_\beta$ (the $k+1$st basic open set in the enumeration) and $\sigma(x_1, \cdots, x_m)$ (the $k+1$st formula defining a non-open set) are such that

$$(**) \qquad \mathcal{O}^*_\beta \subseteq [\sigma(x_1, \cdots, x_m)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$$

then there is an $\mathcal{O}_\beta \in \mathfrak{q}_m$ where $\mathcal{O}^*_\beta \subseteq \mathcal{O}_\beta \subseteq [\sigma(x_1, \cdots, x_m)]^{(\mathfrak{A},\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\cdots)}$.

We do this as follows. In analogue to Lemma 2.3 to obtain (*) we must have $\langle r_{k+1}, s_{k+1}\rangle \in C = (D \times D) - \Delta$, where

$$D = (A - ([\varphi \vee \psi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)} \cup \{r_1, \cdots, r_k, s_1, \cdots, s_k\})).$$

C is open since $[\varphi \vee \psi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$, and each s_i, r_i, $1 \leqq i \leqq k$ are closed in q_1 and Δ is closed in q_2, because the q_1 topology is Hausdorff.

We obtain (**) as in Lemma 2.3. It can be assumed without loss of generality (by the definition of WT) that $\mathcal{O}^*_\beta = \cap \varphi_{\beta_i} (\Pi^{s_i}_{j=1} B_j)$ and $B_1 = \mathcal{U}$ and $B_2 = A - \mathcal{U}$ and $B_j = \mathcal{O}_j \in q_m$ for $j \geqq 3$. Thus if

$$\cap \varphi_{\beta_i}\left(C \times \prod_{j=3}^{s_i} \mathcal{O}_j \right) - [\sigma(x_1, \cdots, x_m)]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)} \neq \varnothing$$

then we take $\langle r_{k+1}, s_{k+1}\rangle \in C$ so that

$$\cap \varphi_{\beta_i}\left(\langle r_{k+1}, s_{k+1}\rangle \times \prod_{i=k+1}^{s} \mathcal{O}_i \right) - [\sigma(x_1, \cdots, x_m)]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)} \neq \varnothing.$$

Otherwise we let $\langle r_{k+1}, s_{k+1}\rangle$ be an arbitrary member of C.

To finish the proof of the lemma we let $\mathcal{U} = \{r_i\}_{i\in\omega} \cup [\psi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$. Then we claim that $\mathcal{U}$ and $A - \mathcal{U}$ separate $[\psi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$ and $[\varphi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$. This is straightforward, since the sequences $\{r_i\}_{i\in\omega}$ and $\{s_i\}_{i\in\omega}$ were picked to miss both $[\psi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$ and $[\varphi]^{(\mathfrak{A}, q_1, q_2, q_3, \cdots)}$. Also by (*) and the fact that $C \cap \Delta = \varnothing$ we know that $\{r_i\}_{i\in\omega} \cap \{s_i\}_{i\in\omega} = \varnothing$.

Let $(\mathfrak{A}, q^*_1, q^*_2, q^*_3, \cdots)$ be the model formed from $\mathcal{U}, A - \mathcal{U}$ and $(\mathfrak{A}, q_1, q_2, q_3, \cdots)$ as in Lemma 2.2. We will show that

$$(\mathfrak{A}, q^*_1, q^*_2, q^*_3, \cdots) \underset{L(Q^n_{n\in\omega})(A)}{\equiv} (\mathfrak{A}, q_1, q_2, q_3, \cdots).$$

To show this we use induction on the Q^m quantifiers. Suppose we are given $Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m)$. Then since $q_i \subseteq q^*_i$ for all i we have that if

$$(\mathfrak{A}, q_1, q_2, q_3, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m)$$

then

$$(\mathfrak{A}, q^*_1, q^*_2, q^*_3, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m).$$

To show the converse suppose that

$$(\mathfrak{A}, q_1, q_2, q_3, \cdots) \models \sim Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m)$$

and

$$(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots) \models Q^m x_1, \cdots, x_m \chi(x_1, \cdots, x_m).$$

Thus

$$[\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)} = [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)}$$

$$= \bigcup_{\beta \in D} \mathcal{O}_\beta^*,$$

$\mathcal{O}_\beta^*$ a basic open set. By (**), for each $\beta \in D$ we have

$$\mathcal{O}_\beta^* \subseteq \mathcal{O}_\beta \subseteq [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}$$

and $\mathcal{O}_\beta \in \mathfrak{q}_m$. Hence

$$\bigcup_{\beta \in D} \mathcal{O}_\beta = \bigcup_{\beta \in D} \mathcal{O}_\beta^* = [\chi(x_1, \cdots, x_m)]^{(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)}.$$

This is a contradiction so we are done.

We now have enough machinery to prove the main theorem of this section.

THEOREM 3.1.6. *Let Σ be an $L(Q^n_{n\in\omega})$ theory and κ an infinite regular cardinal. Then Σ is consistent with* B0, $\cdots$, B7, $B8_{\varphi_\alpha}$, $\alpha \in I$, *and Q^2xy $(x \neq y)$ if and only if Σ has a 0-dimensional normal complete topological model of cardinality κ where each φ_α, $\alpha \in I$, is continuous (complete in the model theoretic sense).*

PROOF. (if direction) Easy since normal implies Hausdorff.

(only if direction) This proof is analogous to the proof of Theorem 3.1.2 of Sgro [12].

Let $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots)$ be a topological model of B0, $\cdots$, B7, $B8_{\varphi_\alpha}$, $\alpha \in I$, and Q^2xy $(x \neq y)$, where the $\mathfrak{q}_i$ are generated by the definable open sets. By applying Lemma 3.1.5, Theorem 1.4 (union of elementary chains), and Theorem 2.4 κ times we obtain a regular 0-dimensional complete-topological model, $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$. (Note that our procedure does not work for pairs of closed sets.)

Since $(\mathfrak{A}^*, \mathfrak{q}^*)$ is complete-topological and regular we can expand it by adding new function and predicate symbols from a new language, L^*, such that

$$(\mathfrak{A}^*, f^\varphi_{\varphi \in L^*(Q^n_{n\in\omega})}, V^\varphi_i(z_i, x_i, y_1, \cdots, y_{n_i})_{\varphi \in L^*(Q^n_{n\in\omega})}, U^{\mathfrak{A}^*}(x, y), \mathfrak{q}^*)$$

models T, $\forall x Q y U(x, y)$, $\forall x Q y \sim U(x, y)$, for each ψ, $\varphi \in L^*(Q^n_{n\in\omega})$

$$\forall y (Qx\psi(x) \wedge \psi(y) \rightarrow \exists z (U(z, y) \wedge \forall x (U(z, x) \rightarrow \psi(x))),$$

and ψ^φ and φ^V (as in Theorem 2.8). The ψ^φ and φ^V's guarantee that the weak model is a complete topological model. The other formulas state that $U(x, y)$ defines a collection of clopen sets which insure that the topology is 0-dimensional and regular.

We will now define an elementary chain of $L^*(Q^n_{n\in\omega})$ complete topological models, $(\mathfrak{B}_\beta, r_\beta)$, $\beta < \kappa$, as follows:

If $\alpha = 0$ then $(\mathfrak{B}_0, r_0) = (\mathfrak{A}^*, f^\varphi, V^\varphi, U^{\mathfrak{A}}, \mathfrak{q}^*)$ as above.

If $\alpha = \beta + 1$ then we define a theory T_α to be:

$$T_\alpha = \begin{cases} \mathrm{Th}((\mathfrak{B}_\beta, r_\beta)) \\ \sim\varphi(c^1_\varphi, \cdots, c^m_\varphi, b_1, \cdots, b_k) \text{ where} \\ (\mathfrak{B}_\beta, r_\beta) \models\ \sim Q^m x_1, \cdots, x_m\, \varphi(x_1, \cdots, x_m, b_1, \cdots, b_k) \\ \psi(c^1_\varphi, \cdots, c^m_\varphi) \text{ where} \\ \langle f^\varphi_1(b_1, \cdots, b_k), \cdots, f^\varphi_m(b_1, \cdots, b_k)\rangle \in [\psi(x_1, \cdots, x_m)]^{(\mathfrak{B}_\beta, r_\beta)} \text{ and} \\ (\mathfrak{B}_\beta, r_\beta) \models Q^m x_1, \cdots, x_m\, \psi(x_1, \cdots, x_m). \end{cases}$$

T_α is consistent since if

$$\langle f^\varphi_1(b_1, \cdots, b_k), \cdots, f^\varphi_m(b_1, \cdots, b_k)\rangle \in [\psi_i(x_1, \cdots, x_m)]^{(\mathfrak{B}_\beta, r_\beta)}$$

for $0 \leqq i \leqq n$ then

$$\langle f^\varphi_1(b_1, \cdots, b_k), \cdots, f^\varphi_m(b_1, \cdots, b_k)\rangle \in \bigcap_{0 \leq i \leq n} [\psi_i(x_1, \cdots, x_m)]^{(\mathfrak{B}_\beta, r_\beta)}.$$

Thus

$$\bigcap_{0 \leq i \leq n} [\psi_i(x_1, \cdots, x_m)]^{(\mathfrak{B}_\beta, r_\beta)} \not\subseteq [\varphi(x_1, \cdots x_m, b_1 \cdots b_k)]^{(\mathfrak{B}_\beta, r_\beta)}.$$

Take $(\mathfrak{B}_\alpha, r_\alpha)$ to be a model of T_α of cardinality κ, where r_α is the set of definable open sets. The purpose of $(\mathfrak{B}_\alpha, r_\alpha)$ is to enable us to take infinite intersections of open sets and to make them open.

If α is a limit ordinal then we take $(\mathfrak{B}_\alpha, r_\alpha)$ to be the union of the elementary chain $(\mathfrak{B}_\beta, r_\beta)$, $\beta < \alpha$.

Let $(\mathfrak{B}, r)$ be the union of the elementary chain $(\mathfrak{B}_\alpha, r_\alpha)$, $\alpha < \kappa$. By Theorem 1.4 and an easy observation

$$(\mathfrak{B}_\alpha, r_\alpha) \underset{L^*(Q^n_{n\in\omega})}{<} (\mathfrak{B}, r) \quad \text{for} \quad \alpha < \kappa.$$

Define r^* to be the topology generated by $\{\mathcal{O}_{(b,\beta)} \mid b \in B,\ \beta < \kappa\}$, where

$\mathcal{O}_{(b,\beta)} = \bigcap_{\mathcal{O} \in \varepsilon(b,\beta)} \mathcal{O}$ and $\varepsilon(b, \beta) = \{\mathcal{O} \subseteq B \mid b \in \mathcal{O}$ and $\mathcal{O}$ is a definable clopen set of r with parameters from $(\mathfrak{B}_\beta, r_\beta)\}$.

We claim that

$$(\mathfrak{B}, r) \underset{L^*(Q^n_{n\in\omega})(B)}{\equiv} (\mathfrak{B}, r^*).$$

This is most easily shown by induction on the complexity of the formulas with parameters in B. The difficult case is the Q^m clause. Since $r \subseteq r^*$ we have that if $(\mathfrak{B}, r) \models Q^m x_1, \cdots, x_m \varphi(x_1, \cdots, x_m)$ then $(\mathfrak{B}, r^*) \models Q^m x_1, \cdots, x_m \varphi(x_1, \cdots, x_m)$. Suppose that $(\mathfrak{B}, r) \models \sim Q^m x_1, \cdots, x_m \varphi(x_1, \cdots, x_m)$. Then if

$$[\varphi(x_1, \cdots, x_m)]^{(\mathfrak{B},r)} = [\varphi(x_1, \cdots, x_m)]^{(\mathfrak{B},r^*)},$$

while

$$(\mathfrak{B}, r^*) \models Q^m x_1, \cdots, x_m \varphi(x_1, \cdots, x_m)$$

we have

$$[\varphi(x_1, \cdots, x_m)]^{(\mathfrak{B},r^*)} = \bigcup_{t\in T} \prod_{i=1}^{s_t} \mathcal{O}(b_i, \beta_i),$$

which implies

$$\langle f_1^\varphi(b_1, \cdots, b_k), \cdots, f_n^\varphi(b_1, \cdots, b_k)\rangle \in \prod_{i=1}^{s_j} \mathcal{O}_{(b_i,\beta_i)}$$

for some $j \in T$. However,

$$\prod_{i=1}^{s_j} \mathcal{O}_{(b_i,\beta_i)} = \bigcap_{\gamma\in G} [\psi_\gamma(x_1, \cdots, x_m)]^{(\mathfrak{B},r)}$$

where $|G| < \kappa$. This follows from the fact that a cartesian product of intersections is the intersection of cartesian products and that a finite cartesian product of definable sets is definable.

Thus for some $\theta < \kappa$, θ a sufficiently large limit ordinal, we have from the definition of T_α, $\alpha < \kappa$, that

$$(\mathfrak{B}_\theta, r_\theta) \models \sim \varphi(c^1_\varphi, \cdots, c^m_\varphi) \wedge \bigwedge_{\gamma\in G} \psi_\gamma(c^1_\varphi, \cdots, c^m_\varphi).$$

Hence $(\mathfrak{B}, r^*) \models \sim Q^m x_1, \cdots, x_m \varphi(x_1, \cdots, x_m)$ which is a contradiction.

$(\mathfrak{B}, r^*)$ is 0-dimensional and regular, since $(\mathfrak{B}, r^*)$ has a clopen basis of cardinality κ which is closed under intersections of cardinality less than κ. This is because κ is regular, the definition of $T_\alpha, \alpha < \kappa$, and r^*. We will show $(\mathfrak{B}, r^*)$ is normal by using a generalization of [2, Theorem 18.14, p. 121] as follows:

THEOREM 3.1.7. *Let (X, τ) be a regular topological space of cardinality κ, κ regular. Then if τ has a basis of cardinality κ closed under intersections of cardinality less than κ then it is normal. In fact it is paracompact* [2, p. 338].

Also note that each φ_α is continuous in $(\mathfrak{B}, r^*)$ since $\varphi_\alpha^{-1}(\bigcap_{\beta\in D}\mathcal{O}_\beta) = \bigcap_{\beta\in D}\varphi_\alpha^{-1}(\mathcal{O}_\beta)$.

One should notice that in the above proof we have actually constructed $(\mathfrak{B}, r^*)$ so that each product topology is normal.

This theorem has the following interesting corollary.

COROLLARY 3.1.8. *Let Σ be a countable $L(Q^n_{n\in\omega})$ theory. Then Σ is consistent with B0, $\cdots$, B7, $\mathrm{B8}_{\varphi_\alpha}$, $\alpha \in I$, and $Q^2xy\ (x \neq y)$ if and only if Σ has a second countable 0-dimensional metrizable complete topological model where each φ_α is continuous.*

PROOF. Use the fact that a second countable, regular and Hausdorff space is metrizable.

§3.2. In this section we study the interrelation of $L(Q)$ theories and $L(Q^n_{n\in\omega})$ theories. The reason for this is that in $L(Q^n_{n\in\omega})$ we have a method of expressing the fact that a function is continuous in a product topology. In mathematics there are many occasions where this situation arises in a first order theory, e.g. topological groups, topological vector spaces, etc. It is natural to ask what conditions on functions (or relations) in an $L(Q)$ theory, Σ, are necessary to insure that they can be interpreted as continuous functions in some $L(Q^n_{n\in\omega})$ theory extending Σ.

The following definition and theorem formalize this.

DEFINITION 3.2.1. A collection of (n, m)-ary relations $\varphi_\alpha(x_1, \cdots, x_n, y_1, \cdots, y_m)$, $\alpha \in I$, is called *$L(Q)$-continuous* (in Σ) if and only if

$$\bigwedge_{i=1}^{m} Qy_i\psi_i(y_i) \rightarrow \forall x_1, \cdots, x_k Qz\left(\exists y_1, \cdots, y_m\left(\bigwedge_{i=1}^{m} \psi_i(y_i) \wedge \theta(t, y_1, \cdots, y_m)\right)\right),$$

$\vec{t} \in (\sigma(\vec{x})/z)$ and $\sigma : n \rightarrow n$, is consistent with Σ (where θ is an arbitrary composition of the φ_α, i.e. $\theta \in WT$, and $(\sigma(\vec{x})/z)$ is the collection of k-tuples which are permuted by σ and then any number of them are replaced by z).

Now we may proceed to prove the main result of this section.

THEOREM 3.2.2. *Let T be an $L(Q)$ theory and $\varphi_\alpha(x_1,\cdots,x_{n_\alpha},y_1,\cdots,y_{m_\alpha})$, $\alpha\in I$, be (n_α,m_α)-ary $L(Q)$-continuous relations. Then there is an $L(Q^n_{n\in\omega})$ theory T^*, such that $T\subseteq T^*$ and* B0, $\cdots$, B7, B8$_{\varphi_\alpha}$, *$\alpha\in I$, are in T^*. (This is to say that we can find a complete topological model $(\mathfrak{A},\mathfrak{q})$ of T where each φ_α is continuous in the product topology.)*

PROOF. Let $(\mathfrak{A},\mathfrak{q})$ be an $L(Q)$ topological model of T where $\mathfrak{q}$ is generated by the definable open sets with parameters (one exists by Theorem 2.1). Define $\mathfrak{q}^*_n$ as follows:

$\mathfrak{q}^*_n$ = the topology generated by

$$\left\{\varphi\left(\prod_{i=1}^{s}\mathcal{O}_i\right)\,\middle|\,\mathcal{O}_i\in\mathfrak{q},\mathcal{O}_i\text{ definable, }\varphi\in WT\text{ and }\varphi\text{ maps into }A^n\right\}.$$

We claim that

$$(\mathfrak{A},\mathfrak{q}^*_1,\mathfrak{q}^*_2,\mathfrak{q}^*_3,\cdots)\underset{L(Q)(A)}{\equiv}(\mathfrak{A},\mathfrak{q}),$$

which implies the theorem, since $(\mathfrak{A},\mathfrak{q}^*_1,\mathfrak{q}^*_2,\mathfrak{q}^*_3,\cdots)$ models B0, $\cdots$, B7, B8$_{\varphi_\alpha}$, $\alpha\in I$, by a slight modification of Lemma 2.2.

We will show that they are $L(Q)$ elementarily equivalent by showing that for every subbasic open set $\mathcal{O}^*\in\mathfrak{q}^*_n$, i.e. $\mathcal{O}^*=\varphi(\Pi_{i=1}^s\mathcal{O}_i)$, $\mathcal{O}_i\in\mathfrak{q}$, $\varphi\in WT$, there is a $\psi(x_1,\cdots,x_n)\in L(Q)$ with parameters from A such that $\mathcal{O}^*=\{\vec{a}\mid(\mathfrak{A},\mathfrak{q})\models\psi[\vec{a}]\}$, and also

$$(*)\qquad (\mathfrak{A},\mathfrak{q})\models\forall x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_nQx_i\,\psi(x_1,\cdots,x_n)$$

for all $1\leqq i\leqq n$.

To show this we take $\varphi\in WT$ (one should notice, since all members of WT_0 are definable and the inductive steps for WT_n are definable, that each $\varphi\in WT$ is definable), then $\varphi(\Pi_{i=1}^s\mathcal{O}_i)$ is definable by

$$\theta(x_1,\cdots,x_n)=\exists y_1,\cdots,y_m\left(\bigwedge_{i=1}^{m}\psi_i(y_i)\wedge\varphi(x_1,\cdots,x_n,y_1,\cdots,y_m)\right)$$

where each ψ_i defines $\mathcal{O}_i$.

To show $(\mathfrak{A},\mathfrak{q})\models\forall x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n\,Qx_i\,\theta(x_1,\cdots,x_n)$ for all $1\leqq i\leqq n$ we use the fact that each member of WT_0 is $L(Q)$-continuous by the hypothesis to the theorem. From this it can be seen that each $\varphi\in WT$ is in fact $L(Q)$-continuous. This is because composition, projection, maps of coordinates and products of functions preserve $L(Q)$-continuity. Then we see that (*) is a consequence of $L(Q)$-continuity and the definition of θ.

Thus to return to showing the $L(Q)$ elementary equivalence of $(\mathfrak{A}, \mathfrak{q})$ and $(\mathfrak{A}, \mathfrak{q}_1^*, \mathfrak{q}_2^*, \mathfrak{q}_3^*, \cdots)$ we notice that (*) implies that in fact $q = q_1^*$. Hence we are done.

Consider a group $(G, \cdot)$. Now take a topology τ on G. We call $(G, \cdot, \tau)$ a *topological group* if $^{-1}$ and $\cdot$ are continuous maps into G. Other definitions of topological-algebraic structures, e.g. a topological vector space, often appear in mathematics. Using Theorem 3.2.2 we are now able to give an $L(Q)$ axiomatization of their $L(Q)$ theories. For more details on topological groups, etc., see [5].

We formalize these comments in the following corollary.

COROLLARY 3.2.3. *Let T be an $L(Q)$ theory. Then T has a topological group model if and only if T is consistent with the basic $L(Q)$ axioms, group axioms and $Qx\varphi(x) \to Qx\varphi(\vec{t})$, where*

$$\vec{t} \in \left(\frac{y_{\sigma(1)}^{\varepsilon(1)} \cdot y_{\sigma(2)}^{\varepsilon(2)} \cdot \cdots \cdot y_{\sigma(k)}^{\varepsilon(k)}}{x} \right),$$

$$\sigma : k \to k \ \textit{and} \ \varepsilon : k \to \{1, -1\}.$$

PROOF. These axioms for topological groups are just the definition of $L(Q)$-continuity for x^{-1} and $\cdot$.

COROLLARY 3.2.4. *Let T be an $L(Q)$ theory. Then T has a topological abelian group model if and only if T is consistent with the basic $L(Q)$ axioms, abelian group axioms, $Qx\varphi(x) \to Qx\varphi(x^{-1})$, $Qx\varphi(x) \to Qx\varphi(x^n \cdot y)$.*

PROOF. Analogous to Corollary 3.2.3.

We will continue the study of the $L(Q)$ theory and decidability of topological abelian groups and vector spaces in Sgro [13].

§3.3. Using the completeness theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ in §1 we can give a completeness theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ which is the infinitary logic formed by combining $L_{\omega_1\omega}$ with the quantifier symbols Q^n, $n \in \omega$. $L_{\omega_1\omega}$ is the infinitary logic formed by allowing countably infinite conjuntions but only finite quantifiers.

In $L_{\omega_1\omega}(Q^n_{n\in\omega})$ the notion of $(\mathfrak{A}, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \cdots) \models \varphi[a_1, \cdots, a_n]$ is defined in the natural way.

The axioms for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ are straightforward and are adaptations to $L_{\omega_1\omega}(Q^n_{n\in\omega})$ of those found in Keisler [7] and Sgro [12].

I. Axioms of $L(Q^n_{n\in\omega})$.

II. $\bigwedge_{n\in\omega}(\varphi \to \psi_n) \to (\varphi \to \bigwedge_{n\in\omega}\psi_n)$.

III. $(\bigwedge_{n\in\omega}\psi_n) \to \psi_m$, $m \in \omega$.

IV. $\bigwedge_{n\in\omega} Q^m x_1, \cdots, x_m \psi_n(x_1, \cdots, x_m) \to Q^m x_1, \cdots, x_m \bigvee_{n\in\omega}\psi_n(x_1, \cdots, x_m)$.

The rules of inference are modus ponens, generalization and the following infinite rule:

$$\text{From } \psi_0, \psi_1, \psi_2, \cdots, \text{ infer } \bigwedge_{n\in\omega} \psi_n.$$

We thus are now able to prove:

THEOREM 3.3.1. *A sentence ψ of $L_{\omega_1\omega}(Q^n_{n\in\omega})$ is consistent with* I, II, III, *and* IV *if and only if ψ has a complete-topological model where each φ_α, $\alpha \in I$, is continuous.*

PROOF. This completeness theorem for $L_{\omega_1\omega}(Q^n_{n\in\omega})$ may be proved as in [12, 3.3.1]. The only observation we need to make is that when we obtain $(\mathfrak{A}, q_1^*, q_2^*, q_3^*, \cdots)$, an $L'(Q^n_{n\in\omega})$ model of B0, $\cdots$, B7 and $B8_{\varphi_\alpha}$, $\alpha \in I$, and ψ, where ψ is equivalent to an $L'(Q^n_{n\in\omega})$ sentence (by Theorem 2.4 and the fact that L is countable), we obtain a topology, r, on $\mathfrak{A}$ such that

$$(\mathfrak{A}, r) \underset{L'(Q^n_{n\in\omega})(A)}{\equiv} (\mathfrak{A}, q_1^*, q_2^*, q_3^*, \cdots).$$

Hence $(\mathfrak{A}, r) \models \psi$ and is complete-topological.

Concluding this paper we note that the counterexamples to definability and interpolation for $L(Q)$ (presented in [12]) also work for $L(Q^n_{n\in\omega})$ using the same arguments.

ACKNOWLEDGEMENTS

The author would especially like to thank his thesis advisor, Professor H. J. Keisler, for his comments, suggestions and guidance during the preparation of this paper. He would also like to express his appreciation to K. Kunen, K. J. Barwise and M. E. Rudin for their welcome advice and assistance.

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